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ON THE GAUSS MAP OF MINIMAL SURFACES IMMERSED IN $\mathbf{R}^n$

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Abstract

In this paper, we prove that the Gauss map of a nonflat complete minimal surface immersed in a Euclidean n-space $\mathbf{R}^n$ can omit at most $n(n+1)/2$ hyperplanes in a complex projective $(n-1)$-space $\mathbf{C}P^{n-1}$ located in general position.

1. Introduction

Let M be a smooth oriented two-manifold without boundary. Take an immersion $f: M \to \mathbf{R}^n$. The metric on M induced from the standard metric ds_E^2 on $\mathbf{R}^n$ by f is denoted by ds^2. Let Δ denote the Laplace-Beltrami operator of (M, ds^2). The local coordinates (x, y) on (M, ds^2) are called *isothermal* if $ds^2 = h(dx^2 + dy^2)$ for some local function $h > 0$. Make M into a Riemann surface by decreeing that the 1-form $dx + idy$ is of type $(1, 0)$, where (x, y) are any isothermal coordinates. In terms of the holomorphic coordinate $z = x + iy$, we can write

$$\Delta = \frac{-4}{h}\frac{\partial^2}{\partial z \partial \bar{z}}.$$

We say that f is *minimal* if $\Delta f = 0$, i.e., an immersion into $\mathbf{R}^n$ is minimal if and only if it is harmonic relative to the induced metric.

The Gauss map of f is defined to be

$$G: M \to \mathbf{C}P^{n-1}, \qquad G(z) = [(\partial f/\partial z)],$$

where $[(\cdot)]$ denotes the complex line in $\mathbf{C}^n$ through the origin and $(\cdot)$. By the assumption of minimality of M, G is a holomorphic map of M into $\mathbf{C}P^{n-1}$.

In 1981, F. Xavier showed that the Gauss map of a nonflat complete minimal surface in $\mathbf{R}^3$ cannot omit seven points of the sphere [15]. In 1988, Fujimoto reduced seven to five, which is sharp [6]. For the $n > 3$

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case, Fujimoto [7] proved that the Gauss map G of a complete minimal surface M in $\mathbf{R}^n$ can omit at most $n(n+1)/2$ hyperplanes in general position, provided G is nondegenerate, i.e., $G(M)$ is not contained in any hyperplane in $\mathbf{C}P^{n-1}$.

In this paper, we will remove Fujimoto's "nondegenerate" condition. The map G is called *k-nondegenerate* if $G(M)$ is contained in a k-dimensional linear subspace of $\mathbf{C}P^{n-1}$, but none of lower dimension. We shall give the following theorem.

Theorem 1. *Let M be a nonflat complete minimal surface immersed in $\mathbf{R}^n$ and assume that the Gauss map G of M is k-nondegenerate $(0 \le k \le n-1)$. Then G can omit at most $(k+1)(n-k/2-1)+n$ hyperplanes in $\mathbf{C}P^{n-1}$ located in general position.*

In particular, we have

Corollary. *Let M be a nonflat complete minimal surface immersed in $\mathbf{R}^n$. Then the Gauss map G can omit at most $n(n+1)/2$ hyperplanes in $\mathbf{C}P^{n-1}$ located in general position.*

Proof. We can assume G is k-nondegenerate $(0 \le k \le n-1)$, because for $0 \le k \le n-1$, we have:

$$n(n+1)/2 \ge (k+1)(n-k/2-1)+n.$$

Thus the theorem implies the corollary.

2. Basic concepts of holomorphic curves into projective spaces

In this section, we shall recall some known results in the theory of holomorphic curves in $\mathbf{C}P^n$.

(A) Associated curve. Let f be a k-nondegenerate holomorphic map of $\Delta_R := \{z; |z| < R\}$ $(\subset C)$ into $\mathbf{C}P^n$, where $0 < R \le +\infty$. Since $f(\Delta_R)$ is contained in a k-dimensional subspace of $\mathbf{C}P^n$, we may assume that $f(\Delta_R)$ is contained in $\mathbf{C}P^k$, so that $f: \Delta_R \to \mathbf{C}P^k$ is nondegenerate. Take a reduced representation $f = [Z_0 : \cdots : Z_k]$, where $Z = (Z_0, \cdots, Z_k): \Delta_R \to C^{k+1} - \{0\}$ is a holomorphic map. Denote $Z^{(j)}$ the j th derivative of Z and define

$$\Lambda_j = Z^{(0)} \wedge \cdots \wedge Z^{(j)}: \Delta_R \to \bigwedge^{j+1} C^{k+1}$$

for $0 \le j \le k$. Evidently $\Lambda_{k+1} \equiv 0$.

Denote

$$P: \bigwedge^{j+1} C^{k+1} - \{0\} \to P\left(\bigwedge^{j+1} C^{k+1}\right) = \mathbf{C}P^{N_j},$$

where $N_j = \binom{k+1}{j+1} - 1$, and P is the natural projection. Λ_j projects down to a curve

$$f_j = P(\Lambda_j) : \Delta_R \to \mathbf{C}P^{N_j}, \qquad 0 \leq j \leq k,$$

called the *jth associated curve* of f. Let ω_j be the Fubini-Study form on $\mathbf{C}P^{N_j}$, and

$$\Omega_j = f_j^* \omega_j, \qquad 0 \leq j \leq k, \tag{2.1}$$

be the pullback via the jth associated curve. It is well known [4] (see also [12]) that, in terms of the homogeneous coordinates,

$$\Omega_j = f_j^* \omega_j = dd^c \log |\Lambda_j|^2 = \frac{i}{2\pi} \frac{|\Lambda_{j-1}|^2 |\Lambda_{j+1}|^2}{|\Lambda_j|^4} dz \wedge d\bar{z} \tag{2.2}$$

for $0 \leq j \leq k$, and by convention $\Lambda_{-1} \equiv 1$. Note that $\Omega_k \equiv 0$. It follows that

$$\operatorname{Ric} \Omega_j = \Omega_{j-1} + \Omega_{j+1} - 2\Omega_j .$$

(B) Projective distance. For integers $1 \leq q \leq p \leq n+1$, the interior product of vectors $\xi \in \bigwedge^{p+1} C^{k+1}$ and $\alpha \in \bigwedge^{q+1} C^{k+1^*}$ is defined by

$$(\xi \llcorner \alpha, \beta) = (\xi, \alpha \wedge \beta) = (\alpha \wedge \beta)(\xi)$$

for any $\beta \in \bigwedge^{p-q} C^{k+1^*}$. For $x \in P(\bigwedge^{p+1} C^{k+1})$ and $a \in P(\bigwedge^{q+1} C^{k+1^*})$ the projective distance $\|x, a\|$ is defined by

$$\|x, a\| = \frac{|\xi \llcorner \alpha|}{|\xi||\alpha|},$$

where $\xi \in \bigwedge^{p+1} C^{k+1} - \{0\}$ and $\alpha \in \bigwedge^{q+1} C^{k+1^*} - \{0\}$; $P(\xi) = x$ and $P(\alpha) = a$.

For a hyperplane a of $\mathbf{C}P^k$, denote

$$f_j \llcorner a = P(\Lambda_j \llcorner \alpha) : \Delta_R \to P\left(\bigwedge^j C^{k+1}\right), \tag{2.4}$$
$$P(\Lambda_j) = f_j, \ P(\alpha) = a,$$

and

$$\varphi_j(a) = \|f_j, a\|^2. \tag{2.5}$$

Note that $0 \leq \varphi_j(a) \leq \varphi_{j+1}(a) \leq 1$ for $0 \leq j \leq k$, and $\varphi_k(a) \equiv 1$.

We need the following well-known lemma (see [4], [12], or [14]).

Lemma 2.1. *Let a be a hyperplane in* $\mathbf{C}P^k$. *Then for any constant* $N > 1$ *and* $0 \le p \le k-1$,

$$dd^c \log \frac{1}{N - \log \phi_p(a_j)} \ge \left\{ \frac{\phi_{p+1}(a_j)}{\phi_p(a_j)(N - \log \phi_p(a_j))^2} - \frac{1}{N} \right\} \Omega_p \tag{2.6}$$

on $\Delta_R - \{\phi_p = 0\}$.

(C) Nochka weight and product to sum estimate. Let $H_1, \cdots, H_q$ be the hyperplanes in $\mathbf{C}P^n$ in general position. Then H_i can be considered as a point in $\mathbf{C}P^{n^*}$, where $\mathbf{C}P^{n^*}$ is the dual space of $\mathbf{C}P^n$. Let $l: \mathbf{C}P^k \to \mathbf{C}P^n$ be the inclusion map. Then the dual map $l^*: \mathbf{C}P^{n^*} \to \mathbf{C}P^{k^*}$ is surjective. Let $a_i = l^*(H_i)$. According to Chen [2], we define the concept of *n-subgeneral position* here.

Definition 2.1. The hyperplanes $a_1, \cdots, a_q$ in $\mathbf{C}P^k$ are called in *n-subgeneral position* iff for every injective map $\lambda: Z[0, n] \to Z[1, q]$, there are $\alpha_{\lambda(i)} \in C^{k+1^*} - \{0\}$ such that $a_{\lambda(i)} = P(\alpha_{\lambda(i)})$ for $i = 0, 1, \cdots, n$ and such that the vectors $\alpha_{\lambda(0)}, \cdots, \alpha_{\lambda(n)}$ generate C^{k+1^*}.

It is easy to check that if $H_1, \cdots, H_q$ are in general position in $\mathbf{C}P^n$, then $a_1, \cdots, a_q$ are in n-subgeneral position in $\mathbf{C}P^k$.

We have the following lemma.

Lemma 2.2 (See Chen [2, Theorem 6.16], also Nochka [8]). *Let $a_1, \cdots, a_q$ be hyperplanes in $\mathbf{C}P^k$ in n-subgeneral position. Then there exist a function $\omega : Q \to R(0, 1]$ and a number $\theta > 0$ with the following properties*:

(1) $0 < \omega(j)\theta \le 1$ *for all* $j \in Q$.

(2) $q - 2n + k - 1 = \theta(\sum_{j=1}^q \omega(j) - k - 1)$.

(3) $1 \le (n+1)/(k+1) \le \theta \le (2n - k + 1)/(k+1)$.

We will call ω the *Nochka weight* for hyperplanes $\{a_i\}$.

We also have the product-to-sum estimate as follows:

Lemma 2.3 (See Chen [2, Theorem 7.3]). *Suppose the above assumptions are true, and take $p \in Z[0, k-1]$. Then for any constant $N \ge 1$, $1/q \le \lambda p \le 1/(k-p)$, there exists a positive constant $C_p > 0$ which only depends on p and the given hyperplanes such that*

$$\begin{aligned} C_p \left(\prod_{j=1}^q \left(\frac{\phi_{p+1}(a_j)}{\phi_p(a_j)} \right)^{\omega(j)} \frac{1}{(N - \log \phi_p(a_j))^2} \right)^{\lambda p} \\ \le \sum_{j=1}^q \frac{\phi_{p+1}(a_j)}{\phi_p(N - \log \phi_p(a_j))^2} \end{aligned} \tag{2.7}$$

on $\Delta_R - \{\phi_p = 0\}$.

3. The main lemma

In this section, we retain the notation of §2. For hyperplanes $a_1, \cdots, a_q$ in $\mathbf{C}P^k$, let ω be their Nochka weight (see Lemma 2.2).

Let $\Omega_p = \frac{i}{2\pi} h_p(z)\, dz \wedge d\bar{z}$ and

$$\sigma_p = C_p \prod_j^q \left[\left(\frac{\phi_{p+1}(a_j)}{\phi_p(a_j)} \right)^{\omega(j)} \frac{1}{(N - \log \phi_p(a_j))^2} \right]^{\lambda p} h_p, \tag{3.1}$$

where C_p is the constant in the product-to-sum estimate (cf. Lemma 2.3), $\lambda p = 1/[k - p + 2q(k-p)^2/N]$, and $N \geq 1$.

We take the geometric mean of the σ_p and define

$$\Gamma = \frac{i}{2\pi} c \prod_{p=0}^{k-1} \sigma_p^{\beta_k/\lambda p}\, dz \wedge d\bar{z}, \tag{3.2}$$

where $\beta_k = 1/\sum_{p=0}^{k-1} \lambda p^{-1}$ and $c = 2(\prod_{p=0}^{k-1} \lambda p^{\lambda p^{-1}})^{\beta_k}$. Let

$$\Gamma = \frac{i}{2\pi} h(z)\, dz \wedge d\bar{z}, \qquad \operatorname{Ric}\Gamma = dd^c \ln h(z).$$

Then

$$h(z) = c \prod_{j=1}^q \left(\frac{1}{\phi_0(a_j)^{\omega(j)}} \right)^{\beta_k} \prod_{j=1}^q \left[\prod_{p=0}^{k-1} \frac{h_p^{\beta_k/\lambda p}}{(N - \log \phi_p(a_j))^{2\beta_k}} \right]. \tag{3.3}$$

Lemma 3.1. *For $q \geq 2n - k + 2$, and*

$$\frac{2q}{N} < \frac{\sum_{j-1}^q \omega(j) - (k+1)}{k(k+2)},$$

we have $\operatorname{Ric}\Gamma \geq \Gamma$.

Proof. From (3.3) it follows that

$$\begin{aligned} \operatorname{Ric}\Gamma = & -\beta_k \sum_{j=1}^q \omega(j) dd^c \log \phi_0(a_j) \\ & + \beta_k \sum_{j=1}^q \sum_{p=1}^{k-1} dd^c \log \left(\frac{1}{N - \log \phi_p(a_j)} \right)^2 + \beta_k \sum_{p=0}^{k-1} (1/\lambda p) \operatorname{Ric}\Omega_p. \end{aligned}$$

By Lemma 2.1, (2.3), and that $dd^c \log \phi_0(a_j) = -\Omega_0$, we have

$$\begin{aligned} \operatorname{Ric}\Gamma \geq \beta_k \Bigg(&\sum_{j=1}^{q} \omega(j)\Omega_0 \\ &+ 2\sum_{j=1}^{q}\sum_{p=0}^{k-1} \frac{\phi_{p+1}(a_j)}{\phi_p(a_j)(N - \log\phi_p(a_j))^2}\Omega_p - \frac{2q}{N}\sum_{p=0}^{k-1}\Omega_p \\ &+ \sum_{p=0}^{k-1}\left[(k-p) + (k-p)^2\frac{2q}{N}\right]\{\Omega_{p+1} - 2\Omega_p + \Omega_{p-1}\}\Bigg). \end{aligned} \tag{3.4}$$

Using Lemma 2.3 we obtain

$$\begin{aligned} &\sum_{j=1}^{q} \frac{\phi_{p+1}(a_j)}{\phi_p(a_j)(N - \log\phi_p(a_j))^2}\Omega_p \\ &\quad \geq C_p \left[\prod_{j=1}^{q}\left(\frac{\phi_{p+1}(a_j)}{\phi_p(a_j)}\right)^{\omega(j)} \frac{1}{(N - \log\phi_p(a_j))^2}\right]^{\lambda p} \Omega_p \\ &\quad = \frac{i}{2\pi}\sigma_p\, dz \wedge d\bar{z}. \end{aligned}$$

We also notice that $\Omega_k = 0$, so that

$$\sum_{p=0}^{k-1}(k-p)(\Omega_{p+1} - 2\Omega_p + \Omega_{p-1}) = -(k+1)\Omega_0,$$

and therefore

$$\begin{aligned} \operatorname{Ric}\Gamma \geq \beta_k \Bigg(&\sum_{j=1}^{q}\omega(j)\Omega_0 + 2\frac{i}{2\pi}\sum_{p=0}^{k-1}\sigma_p\, dz\wedge d\bar{z} - (k+1)\Omega_0 \\ &- (k^2 + 2k)\frac{2q}{N}\Omega_0 \\ &+ \sum_{p=1}^{k-2}[(k-p+1)^2 \\ &\qquad -2(k-p)^2 + (k-p-1)^2 - 1]\frac{2q}{N}\Omega_p + \frac{2q}{N}\Omega_{k-1}\Bigg). \end{aligned}$$

We use the following elementary inequality:

For all the positive numbers $x_1, \cdots, x_n$ and $a_1, \cdots, a_n$,

$$a_1x_1 + \cdots + a_nx_n \geq (a_1 + \cdots + a_n)(x_1^{a_1}\cdots x_n^{a_n})^{1/(a_1+\cdots+a_n)}. \tag{3.5}$$

Letting $a_p = \lambda p^{-1}$ in (3.5), we have

$$\sum_{p=0}^{k-1} \sigma_p \geq \frac{c}{2\beta_k} \prod_{p=0}^{k-1} \sigma_p^{\beta_k/\lambda p}$$

and therefore

$$\operatorname{Ric}\Gamma \geq \beta_k \Big[\sum_{j=1}^{q} \omega(j) - (k+1) - (k^2+2k)\frac{2q}{N})\Omega_0 + \sum_{p=1}^{k-2} \frac{2q}{N}\Omega_p + \frac{2q}{N}\Omega_{k-1}\Big] + \Gamma.$$

By Lemma 2.2 we obtain

$$\theta \left(\sum_{j=1}^{q} \omega(j) - k - 1 \right) = q - 2n + k - 1 > 0,$$

and $\theta > 0$, so $\sum_{j-1}^{q} \omega(j) - (k+1) > 0$. Using the assumption of the lemma hence gives $\operatorname{Ric}\Gamma \geq \Gamma$. q.e.d.

By the Schwarz lemma, we have

$$h(z) \leq \left(\frac{2R}{R^2 - |z|^2} \right)^2 . \tag{3.6}$$

Main Lemma. *Let $f = [Z_0 : \cdots : Z_k]\colon \Delta_R \to \mathbf{C}P^k$ be a nondegenerate holomorphic map, $a_0, \cdots, a_q$ be hyperplanes in $\mathbf{C}P^k$ in n-subgeneral position, and $\omega(j)$ be their Nochka weight. Let $P(\alpha_i) = a_i$, where P is a projection, and $Z = (Z_0, \cdots, Z_k)$. If $q > 2n - k + 1$ and*

$$N > \frac{2q(k^2+2k)}{\sum_{j-1}^{q} \omega(j) - (k+1)},$$

then there exists some positive constant C such that

$$|Z|^H \frac{\prod_{p=0}^{k-1} \prod_{j=1}^{q} |\Lambda_p \llcorner \alpha_j|^{4/N} |\Lambda_k|^{1+2q/N}}{\prod_{j=1}^{q} |(Z, \alpha_j)|^{\omega(j)}} \tag{3.7}$$

$$\leq C \left(\frac{2R}{R^2 - |z|^2} \right)^{k(k+1)/2 + \sum_{p=0}^{k-1} (k-p)^2 2q/N},$$

where H is given by $\sum_{j=1}^{q} \omega(j) - (k+1) - (k^2+2k-1)2q/N$.

Proof. We shall calculate $\prod_{p=0}^{k-1} h_p^{1/\lambda p}$. By (2.2), we have

$$h_p^{1/\lambda p} = \left(\frac{|\Lambda_{p-1}|^2 |\Lambda_{p+1}|^2}{|\Lambda_p|^4} \right)^{(k-p)+(k-p)^2 2q/N},$$

so

$$\prod_{p=0}^{k-1} h_p^{1/\lambda p} = |\Lambda_0|^{-2(k+1)-(k^2+2k-1)4q/N} |\Lambda_1|^{8q/N} \cdots |\Lambda_{k-1}|^{8q/N} |\Lambda_k|^{2+4q/N}.$$

Since $|\Lambda_0| = |Z|$ and $\phi_0(a_j) = |(Z, \alpha_j)|^2/|Z|^2$, $\phi_p(a_j) = |\Lambda_p \llcorner \alpha_j|^2/|\Lambda_p|^2$, from (3.3) and (3.6) it follows that

$$|Z|^H \frac{(|\Lambda_1| \cdots |\Lambda_{k-1}|)^{4q/N} |\Lambda_k|^{1+2q/N}}{\prod_{j=1}^q |(Z, \alpha_j)|^{\omega(j)} \left(\prod_{p=0}^{k-1} (N - \log \phi_p(a_j))\right)} < C\left(\frac{2R}{R^2 - |z|^2}\right)^{1/\beta_k}. \tag{3.8}$$

Set $K := \sup_{0<x\le 1} x^{2/N}(N - \log x)$. Since $\phi_p(a_j) < 1$ for all p and j, we have

$$\frac{1}{(N - \log \phi_p(a_j))} \ge \frac{1}{K}\phi_p(a_j)^{2/N} = \frac{1}{K}\frac{|\Lambda_p \llcorner \alpha_j|^{4/N}}{|\Lambda_p|^{4/N}}.$$

Substituting these into (3.8), we obtain the desired conclusion.

4. Proof of the theorem

We will now prove the theorem. The proof basically follows Fujimoto's proof [7].

We may assume M is simply connected, otherwise we consider its universal covering. By Koebe's uniformization theorem, M is biholomorphic to C or to the unit disc Δ. For the case $M = C$, Nochka [8] (see also Chen [2]) proved the Cartan conjecture which implies that a k-nondegenerate holomorphic map from C to $\mathbf{C}P^n$ cannot omit $2n-k+2$ hyperplanes in general position; in this case our theorem is true. For our purpose it suffices to consider the case $M = \Delta$.

Now assume our theorem is not true, namely the Gauss map G omits q hyperplanes $H_1, \cdots, H_q$ in $\mathbf{C}P^{n-1}$ in general position and $q > (k+1)(n-k/2-1)+n$. Let $\omega(j)$ be the Nochka weight of $\{H_i\}$.

Because G is k-nondegenerate, we assume $G(\Delta) \subset \mathbf{C}P^k$, so that $G = [g_0 : \cdots : g_k] : \Delta \to \mathbf{C}P^k$ is nondegenerate. Let $l \colon \mathbf{C}P^k \to \mathbf{C}P^{n-1}$ be the inclusion map, $l^* \colon \mathbf{C}P^{n-1^*} \to \mathbf{C}P^{k^*}$ be the dual map, and $a_i = l^*(H_i)$. Then the $\{a_i\}$ are the hyperplanes in $\mathbf{C}P^k$ in $(n-1)$-subgeneral position.

Let $\widetilde{G} = (g_0, \cdots, g_k)\colon C \to C^{k+1} - \{0\}$; then the metric ds^2 on M induced from the standard metric on $\mathbf{R}^n$ is given by

$$ds^2 = 2|\widetilde{G}|^2|dz|^2. \tag{4.1}$$

By Lemma 2.2, we have

$$q - 2(n-1) + k - 1 = \theta\left(\sum_{j=1}^{q}\omega(j) - k - 1\right),$$

and

$$\theta \le \frac{2(n-1) - k + 1}{k+1} = \frac{2n-k-1}{k+1},$$

so

$$\frac{2\left(\sum_{j=1}^{q}\omega(j) - k - 1\right)}{k(k+1)} = \frac{2(q-2n+k+1)}{\theta k(k+1)} \ge \frac{2(q-2n+k+1)}{(2n-k-1)k} > 1.$$

Consider the numbers

$$\rho = \frac{1}{H}\left[\frac{k}{2}(k+1) + \frac{2q}{N}\sum_{p=0}^{k}(k-p)^2\right], \tag{4.2}$$

$$\gamma = \frac{1}{H}\left[\frac{k}{2}(k+1) + \frac{qk}{N}(k+1) + \frac{2q}{N}\sum_{p=0}^{k-1}p(p+1)\right], \tag{4.3}$$

$$\rho^* = \frac{1}{(1-\gamma)H}. \tag{4.4}$$

Choose some N such that

$$\frac{\sum_{j=1}^{q}\omega(j) - (k+1) - k(k+1)/2}{k^2 + 2k - 1 + \sum_{p=0}^{k}(k-p)^2} > \frac{2q}{N} > \frac{\sum_{j=1}^{q}\omega(j) - (k+1) - k(k+1)/2}{2/q + (k^2+2k-1) + k(k+1)/2 + \sum_{p=0}^{k-1}p(p+1)}$$

so that

$$0 < \rho < 1, \qquad \frac{4\rho^*}{N} > 1. \tag{4.5}$$

Consider the open subset

$$M' = M - \left(\{\widetilde{G}_k = 0\} \bigcup_{1\le j\le q,\, 0\le p\le k} \{\widetilde{G}_p \llcorner \alpha_j = 0\}\right)$$

of M and define the function

$$(4.6) \qquad v = \left(\frac{\prod_{j=1}^{q} |(\widetilde{G}, \alpha_j)|^{\omega(j)}}{\prod_{p=0}^{k-1} \prod_{j=1}^{q} |\widetilde{G}_p \llcorner \alpha_j|^{4/N} |\widetilde{G}_k|^{1+2q/N}} \right)^{\rho^*}$$

on M', where $\widetilde{G}_p = \widetilde{G}^{(0)} \wedge \cdots \wedge \widetilde{G}^{(p)}$ and $P(\alpha_j) = a_j$.

Let $\pi: \widetilde{M}' \to M'$ be the universal covering of M'. Since $\log v \circ \pi$ is harmonic on $\widetilde{M}'$ by the assumption, we can take a holomorphic function β on $\widetilde{M}'$ such that $|\beta| = v \circ \pi$. Without loss of generality, we may assume that M' contains the origin o of C. As in Fujimoto's papers [5], [6], [7], for each point $\tilde{p}$ of $\widetilde{M}'$ we take a continuous curve $\gamma_{\tilde{p}} : [0, 1] \to M'$ with $\gamma_{\tilde{p}}(0) = o$ and $\gamma_{\tilde{p}}(1) = \pi(\tilde{p})$, which corresponds to the homotopy class of $\tilde{p}$. Let $\tilde{o}$ denote the point corresponding to the constant curve o, and set

$$w = F(\tilde{p}) = \int_{\gamma_{\tilde{p}}} \beta(z)\, dz,$$

where z denotes the holomorphic coordinate on M' induced from the holomorphic global coordinate on $\widetilde{M}'$ by π. Then F is a single-valued holomorphic function on $\widetilde{M}'$ satisfying the condition $F(\tilde{o}) = 0$ and $dF(\tilde{p}) \neq 0$ for every $\tilde{p} \in \widetilde{M}'$. Choose the largest R ($\leq +\infty$) such that F maps an open neighborhood U of $\tilde{o}$ biholomorphically onto an open disc Δ_R in C, and consider the map $B = \pi \circ (F \mid U)^{-1} : \Delta_R \to M'$. By the Liouville theorem, $R = \infty$ is impossible.

For each point $a \in \partial\Delta$ consider the curve

$$L_a : w = ta, \qquad 0 \leq t < 1,$$

and the image Γ_a of L_a by B. We shall show that there exists a point a_0 in $\partial\Delta_R$ such that Γa_0 tends to the boundary of M. To this end, we assume the contrary. Then, for each $a \in \partial\Delta_R$, there is a sequence $\{t_v : v = 1, 2, \cdots\}$ such that $\lim_{v\to\infty} t_v = 1$ and $z_0 = \lim_{v\to\infty} B(t_v a)$ exist in M. Suppose that $z_0 \notin M'$. Let $\delta_0 = 4\rho^*/N > 1$. Then obviously,

$$\liminf_{z \to z_0} |\widetilde{G}_k|^{(1+2q/N)\rho^*} \prod_{1\leq j\leq q,\, 1\leq p\leq k-1} |\widetilde{G}_p \llcorner \alpha_j|^{\delta_0} \cdot v > 0.$$

If $\widetilde{G}_k(z_0) = 0$ or $|\widetilde{G}_p \llcorner \alpha_j|(z_0) = 0$ for some p and j, we can find a positive constant C such that $v \geq C/|z - z_0|^{\delta_0}$ in a neighborhood of z_0, and obtain

$$R = \int_{L_a} |dw| = \int_{L_a} \left|\frac{dw}{dz}\right| |dz| = \int v(z)\, |dz| \geq C \int_{\Gamma_a} \frac{1}{|z - z_0|\delta_0} |dz| = \infty.$$

This is a contradiction. Therefore, we have $z_0 \in M'$.

Take a simply connected neighborhood V of z_0, which is relatively compact in M', and set $C' = \min_{z\in V} v(z) > 0$. Then $B(ta) \in V$ $(t_0 < t < 1)$ for some t_0. In fact, if not, Γ_a goes and returns infinitely often from ∂V to a sufficiently small neighborhood of z_0 and so we get the absurd conclusion

$$R = \int_{L_a} |dw| \geq C' \int_{\Gamma_a} |dz| = \infty .$$

By the same argument, we can easily see that $\lim_{t\to 1} B(ta) = z_0$. Since π maps each connected component of $\pi^{-1}(V)$ biholomorphically onto V, there exists the limit

$$\tilde{p}_0 = \lim_{t\to 1}(F \mid U)^{-1}(ta) \in M' .$$

Then $(F \mid U)^{-1}$ has a biholomorphic extension to a neighborhood of a. Since a is arbitrarily chosen, F maps an open neighborhood of $\overline{U}$ biholomorphically onto an open neighborhood of $\overline{\Delta}_R$. This contradicts the property of R. In conclusion, there exists a point $a_0 \in \partial\Delta_R$ such that Γ_{a_0} tends to the boundary of M.

By the definition of $w = F(z)$ we have

$$(4.7)\qquad \left|\frac{dw}{dz}\right| = |\beta|^{1-\gamma}\left|\frac{dw}{dz}\right|^{\gamma} = \left(\frac{\prod_{j=1}^{q} |(\widetilde{G}, \alpha_j)|^{\omega(j)}}{\prod_{p=0}^{k-1}\prod_{j=1}^{q} |\widetilde{G}_p \llcorner \alpha_j|^{4/N} |\widetilde{G}_k|^{1+2q/N}}\right)^{1/H} \left|\frac{dw}{dz}\right|^{\gamma} .$$

Let $Z(w) = \widetilde{G}\circ B(w)$, $Z_0(w) = g_0 \circ B(w), \cdots , Z_k(w) = g_k \circ B(w)$. Since $Z \wedge Z' \wedge \cdots \wedge Z^{(p)} = (\widetilde{G} \wedge \cdots \wedge \widetilde{G}^{(p-1)})(\frac{dz}{dw})^{p(p+1)/2}$, it is easy to see that

$$(4.8)\qquad \left|\frac{dw}{dz}\right| = \left(\frac{\prod_{j=1}^{q} |(Z, \alpha_j)|^{\omega(j)}}{\prod_{p=0}^{k-1}\prod_{j=1}^{q} |\Lambda_p \llcorner \alpha_j|^{4/N} |\Lambda_k|^{1+2q/N}}\right)^{1/H} ,$$

where $\Lambda_p = Z^{(0)} \wedge \cdots \wedge Z^{(p)}$.

On the other hand, the metric in Δ_R induced from $ds^2 = 2|\widetilde{G}|^2 |dz|^2$ through B is given by

$$(4.9)\qquad B^* ds^2 = 2|\widetilde{G}(B(w))|^2 \left|\frac{dz}{dw}\right|^2 |dw|^2 .$$

Combining (4.7) and (4.8) yields

$$B^* \, ds = 2|Z| \left(\frac{\prod_{p=0}^{k-1} \prod_{j=1}^{q} |\Lambda_p \llcorner \alpha_j|^{4/N} |\Lambda_k|^{1+2q/N}}{\prod_{j=1}^{q} |(Z, \alpha_j)|^{\omega(j)}} \right)^{1/H} |dw|.$$

Using the main lemma, we obtain

$$B^* \, ds \leq C \left(\frac{2R}{R^2 - |w|^2} \right)^{\rho} |dw|,$$

where C is a positive constant. Since $\rho < 1$, it then follows that

$$d(0) \leq \int_{\Gamma_{a_0}} ds = \int_{L_{a_0}} B^* \, ds \leq C \int_0^R \left(\frac{2R}{R^2 - |w|^2} \right)^{\rho} |dw| < \infty,$$

where $d(0)$ denotes the distance from the origin o to the boundary of M, contradicting the assumption of completeness of M. Hence the proof of the theorem is complete.

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